STABILITY OF AN ELASTOPLASTIC SPHERE UNDER EXTERNAL PRESSURE

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The stability of ideally plastic, elastoplastic, and reinforced elastoviscoplastic bodies subjected to large subcritical strains was investigated in [1-4]. The problems solved in these papers were related to the stability of systems in which homogeneous stress and strain fields arise in the initial state. The stability of an elastic thick-walled spherical shell subjected to external pressure leading to large subcritical strains was investigated in [5]. The stability of an axisymmetric sphere of elastoplastic material subjected to large plastic strains is examined below.

The system of equations characterizing the behavior of an elastoplastic body subjected to finite strains is taken in the same form as in [4].

For a sphere under external pressure q the relation between the geometric dimensions in the subcritical state, with incompressibility taken into account, is similar to that used in [5]:

$$a=\lambda a^0;\quad a^{03}-a^3=b^{03}-b^3;\quad Q(r)=\frac{r^0}{r}=\left[1+(1-\lambda)^3\left(\frac{a^0}{r}\right)^3\right]^{1/3},$$
$$2\hat{\varepsilon}_1^1=1-Q^{-4};\quad 2\hat{\varepsilon}_2^2=2\hat{\varepsilon}_3^3=1-Q^2;$$
$$\varepsilon_j^i=0;\ i\neq j, \tag{1}$$

where a^0, b^0, r^0, a, b, r are the dimensions of the sphere, cavity, and arbitrary radius within the body before and after deformation; $\hat{\varepsilon}_i^j$ are components of the total strain tensor.

The amplitudes of the stresses σ_i^j and displacements ω^i in the spherical system of coordinates (r, θ, φ) are connected by the relations [4]

$$\sigma_j'^i=g_j^i a_{j\alpha}g^{\alpha\alpha}\nabla_\alpha w_\alpha'+(1-g_j^i)G_{ji}g^{ii}(\nabla_i w_j'+\nabla_j w_i'); \tag{2}$$
$$a_{kj}=\left(r_{kj}-\frac{1}{3}\alpha_j r_{kn}\right)\mu_j;\quad \|r_{in}\|=\|d_{in}\|^{-1};\quad \mu_i=1-2\varepsilon_i^i;$$
$$d_{kj}=\left[\frac{3}{2E_c}+\mu_k\frac{1+\varkappa}{E}\right]g_k^j+\left(\frac{1}{E_k}-\frac{1}{E_c}\right)\frac{e_k^k e_j^j}{e_I^2}+\frac{2(1+\varkappa)}{E}\varepsilon_j^j-\frac{\varkappa}{E}(1-2e_k^k)-\frac{1}{2E_c}; \tag{3}$$

$E_c=\sigma_I/e_I$, $E_k=d\sigma_I/de_I$; $G_{ij}=\mu_i\Big/\left(\frac{3}{2E_c}+\frac{2(1+\varkappa)}{E}\mu_i\right)$, $\alpha_k=1-\mu_{k+1}\mu_{k-1}$, where g^{ij} are the metric tensor components; e_i^j is the strain tensor deviator; e_I and σ_I are the intensities of the strain and stress tensors; E is Young's modulus; $\varkappa$ is Poisson's ratio.

In the case of an incompressible material ($g^0\hat{g}^{-1}=1$) we have the following relation for the increments:

$$\nabla_k w'^k=0. \tag{4}$$

We can then put the relation between the stress and displacement variations in the form

$$\sigma_i'^j=g_i^j\bar{a}_{i\alpha}g^{\alpha\alpha}\nabla_\alpha w_\alpha'+(1-g_i^j)\bar{G}_{ij}g^{jj}(\nabla_i w_j+\nabla_j w_i)+g_i^j b_i p';$$
$$\bar{a}_{kj}=\left(\bar{r}_{kj}-\frac{1}{3}\bar{r}_{km}\right)\mu_j;\quad \|\bar{r}_{in}\|=\|\bar{d}_{in}\|^{-1};\quad \bar{b}_k=r_{km}\bar{B}_m; \tag{5}$$
$$\bar{d}_{mk}=\bar{B}_m g_k^m+\left(\frac{1}{E_k}-\frac{1}{E_c}\right)\frac{e_m^m e_k^k}{e_I^2}+\frac{\varepsilon_k^k}{E};\ \bar{G}_{ij}=\frac{\mu_i}{2\bar{B}_i};\ \bar{B}_j=\frac{3}{2}\left(\frac{1}{E_c}+\frac{1}{E}\mu_j\right),$$

where p' is the hydrostatic pressure.

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Since problem (1) is axisymmetric, we find from (3) that

$$a_{12}=a_{13};\ a_{22}=a_{33};\ a_{21}=a_{31};\ a_{32}=a_{23};\ G_{12}=G_{13};$$
$$G_{21}=G_{31}=G_{23}=G_{32};\ 2G_{32}=a_{33}-a_{32};\ \overline{b}_2=\overline{b}_3. \tag{6}$$

Equalities (6) are valid of they are regarded as having a negative sign.

Substituting relations (2) in the equilibrium equation [5] for the disturbances, we arrive at a system of three second-order differential equations in the displacements $w_1'=u$, $w_2'=v$, $w_3'=w$;

$$a_{11}r^2u_{,rr}+G_{12}u_{,\theta\theta}+\frac{1}{\cos^2\theta}u_{,\varphi\varphi}+L_1ru_{,r}-G_{12}\operatorname{tg}\theta u_{,\theta}+2L_5u+$$
$$+F_{12}v_{,r\theta}-F_{12}\operatorname{tg}\theta v_{,r}+\frac{1}{r}L_2v_{,\theta}-\frac{1}{r}L_2\operatorname{tg}\theta v+\frac{1}{\cos^2\theta}F_{12}w_{,r\varphi}+\frac{1}{r\cos^2\theta}L_2w_{,\varphi}=r^2\rho\ddot{u}; \tag{7}$$

$$F_{21}r^2u_{,r\theta}+L_4ru_{,\theta}+G_{21}r^2v_{,rr}+a_{22}v_{,\theta\theta}+\frac{1}{\cos^2\theta}G_{21}v_{,\varphi\varphi}+$$
$$+L_3rv_{,r}-a_{22}\operatorname{tg}\theta v_{,\theta}-(2L_3+2G_{21}-a_{23}+a_{22}\operatorname{tg}^2\theta)v+\frac{1}{\cos^2\theta}F_{23}w_{,\varphi\theta}+\frac{2}{\cos^2\theta}a_{22}\operatorname{tg}\theta w_{,\varphi}=r^2\rho\ddot{v}; \tag{8}$$

$$F_{21}r^2u_{,r\varphi}+L_4ru_{,\varphi}+F_{23}v_{,\varphi\theta}-F_{22}\operatorname{tg}\theta v_{,\varphi}+G_{21}r^2w_{,rr}+$$
$$+G_{21}w_{,\theta\theta}+\frac{1}{\cos^2\theta}a_{33}w_{,\varphi\varphi}+L_3rw_{,r}+G_{21}\operatorname{tg}\theta w_{,\theta}-2L_3w=r^2\rho\ddot{w}; \tag{9}$$

$$L_1=ra_{11,r}+2(a_{11}+a_{12}-a_{21}+\sigma);\quad L_2=ra_{12,r}-(2G_{12}+a_{22}+$$
$$+a_{32}+2\sigma);\ L_3=rG_{21,r}+G_{12}-G_{21}+\sigma;\quad L_4=L_3+2G_{21}+$$
$$+a_{22}+a_{23};\ L_5=L_2+F_{12}+G_{12};\ F_{ij}=a_{ij}+G_{ij}+\sigma_i^i-\sigma_j^j;\ \sigma=\sigma_1^1-\sigma_2^2. \tag{10}$$

In the case of an incompressible material we obtain in a similar way from (4) and (5) a system for four differential equations for the unknowns u, v, w, and p. The form of the three differential equilibrium equations is similar to (7)-(10) if we replace a_{ij} in them by $\overline{a}_{ij}$, G_{ij} by $\overline{C}_{ij}$ and add to the left-hand side of (7)-(9), respectively, the terms

$$\overline{b}_1r^2p'_{,r}+[2(\overline{b}_1-\overline{b}_2)r+r^2\overline{b}_{1,r}]p';\quad \overline{b}_2r^2p'_{,\theta};\quad \overline{b}_3r^2p'_{,\varphi},$$

and the fourth equation (incompressibility condition) has the form

$$r^2u_{,r}+v_{,\theta}+(1/\cos^2\theta)w_{,\varphi}+2ru-\operatorname{tg}\theta v=0.$$

The boundary conditions are written in the form

$$\sigma_k'^1=0\quad\text{for}\quad r=a;\ b\quad (k=1,2,3). \tag{11}$$

For simplification we consider a static formulation of the problem. We regard the state corresponding to loss of stability as a state for which problem (7)-(9), (11) has a nontrivial, as well as a trivial, solution.

We seek a solution of Eqs. (7)-(9), (11) in the form of a series of spherical functions $Y_{nj}(\theta,\varphi)$;

$$u=\sum_{n=j}^{\infty}\sum_{j=1}^{\infty}A_{nj}(r)Y_{nj}(\theta,\varphi);\quad v=\sum_{n=j}^{\infty}\sum_{j=1}^{\infty}B_{nj}(r)\frac{D}{D\theta}Y_{nj}(\theta,\varphi);$$
$$w=\sum_{n=j}^{\infty}\sum_{j=1}^{\infty}C_{nj}(r)\frac{D}{D\varphi}Y_{nj}(\theta,\varphi);\quad p'=\sum_{n=j}^{\infty}\sum_{j=1}^{\infty}D_{nj}(r)Y_{nj}(\theta,\varphi). \tag{12}$$

From Eqs. (8) and (9), using (12), we can establish, as in [5], the equalities $B_{nj}(r)=C_{nj}(r)$. This is also true when a_{ij} is replaced by $\overline{a}_{ij}$ and G_{ij} by $\overline{G}_{ij}$. The boundary-value problem can be written in terms of the functions A_{nj}, B_{nj}, and D_{nj}. From the system of equations (7)-(9) we obtain two second-order differential equations for functions A_{nj} and B_{nj} (the subscripts n and j are dropped):

$$a_{11}r^2A_{,rr}+L_1rA_{,r}+(2L_5-NG_{12})A-NF_{12}rB_{,r}-NL_2B=0; \tag{13}$$
$$F_{21}r^2A_{,r}+L_4rA+G_{21}r^2B_{,rr}+L_3rB_{,r}-(2L_3+Na_{22})B=0, \tag{14}$$
$$N=n+n^2$$

with boundary conditions

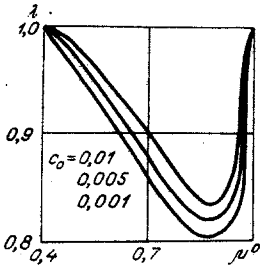

Fig. 1

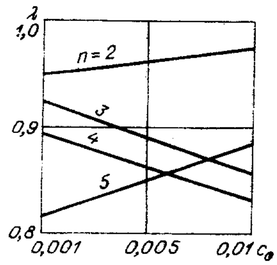

Fig. 2

$$a_{11}r^2A_{,rr} + 2a_{12}rA + Na_{13}B = 0;\ rA + rB = 0 \text{ for } r = a;\ b. \tag{15}$$

For a compressible material, eliminating functions B_{nj} and D_{nj}, we arrive at a fourth-order differential equation for the function $A_{nj}(r)$,

$$\sum_{i=1}^{4} P_i \frac{d^iA}{dr^i} + P_0A = 0, \tag{16}$$

where

$$P_i = \varkappa_i + \beta_i + \gamma_i\ (i = 0, \ldots, 4);\quad \beta_4 = \overline{b}_1 a_3^1;\quad \beta_0 = \overline{b}_1 a_0^1;$$

$$\beta_i = \overline{b}_1 \left(a_i^1 r + a_{i-1}\right) (i = 1, 2, 3);\quad \varkappa_i = a_i^1\left[\overline{b}_{1,r} + \frac{2}{r}\left(\overline{b}_1 - \overline{b}_2\right)\right]$$

$$(i = 0, \ldots, 4);$$

$$\gamma_3 = \gamma_4 = 0;\quad \gamma_2 = -N\left(\overline{a}_{11} - \overline{F}_{12}\right);\quad \gamma_1 = -N\frac{1}{r}\left(\overline{L}_1 - \overline{L}_2 - 4\overline{F}_{12}\right);$$

$$\gamma_0 = N(N-2)\overline{G}_{12}\frac{1}{r^2};\quad a_4^1 = 0;\quad a_3^1 = r^2\overline{b}_3^{-1}\overline{G}_{21};$$

$$a_2^1 = r\left(\overline{L}_3 + 6\overline{G}_{21}\right)/\overline{b}_3;$$

$$a_1^1 = [N(\overline{F}_{21} - \overline{a}_{22}) + 2\overline{L}_3 + 6\overline{G}_{21}]/\overline{b}_3;\quad a_0^1 = \overline{L}_3(N-2)/\overline{b}_3,$$

and boundary conditions of the form

$$r^2A_{,rr} + 2rA_{,r} + (N-2)A = 0,\ a_3^1A_{,rrr} + a_2^1A_{,rr} + \left(a_1^1 + N\frac{\overline{a}_{11} - \overline{a}_{12}}{\overline{b}_1}\right)A_{,r} + a_0^1A = 0. \tag{17}$$

Thus, the problem of determining the critical strain was reduced to the solution of two second-order differential equations (13) and (14) with boundary conditions (15), and in cases of compressible and incompressible materials to the solution of the differential equation (16) with boundary conditions (17). These boundary-value problems are eigenvalue problems. The coefficients of the differential equations depend on the parameters of the medium, design, value of n, and the strain. For fixed values of the parameters and the number n the solution of the problem will give the required value of the critical strain.

The problem was treated numerically for an incompressible material. We used the method of finite differences in the calculations.

As Fig. 1 indicates, for a solid sphere ($b_0 \to 0$) there is no loss of stability, which is consistent with general physical ideas. When $b_0 = a_0$ (in the case of a plastic spherical film) a loss of stability is possible at minimum strains. The calculations were made for $\sigma_T^0 = \sigma_T G^{-1} = -2_{10} - 3$, where σ_T is the yield stress, G is the shear modulus, and different values of $c_0 = cG^{-1}$, where c is the reinforcement coefficient. The relation between σ_I and e_I was selected in the form $\sigma_I = \sigma_T + ce_I$.

Figure 2 shows the critical strain λ as a function of the reinforcement coefficient c_0 for different yield stresses $\sigma_T^0 = -2_{10} - 3$ and $\mu^0 = b^0/a^0 = 0.5$. It is obvious that n = 2 corresponds to the minimum value of the critical strain (and, hence, force), at which loss of stability occurs.

LITERATURE CITED

1. S. Zahorski, "On plastic instability in some cases of simple flow," Bull. Acad. Polon. Sci., Ser. Sci. Tech., 12, No. 12 (1964).
2. S. Zahorski, "Kinematic stability in the case of slow steady plastic flow," Arch. Mech. Stosow., 16, No. 6 (1964).

3. A. N. Sporykhin and V. G. Trofimov, "Plastic instability in some cases of simple flow," Prikl. Mat. Mekh., 38, No. 4 (1974).
4. A. V. Skachenko and A. N. Sporykhin, "Stability of elastroplastic bodies under large plastic deformations," Prikl. Mekh., 12, No. 5, 11-17 (1976).
5. Z. Wesolowski, "Stability of an elastic thick-walled spherical shell loaded by an external pressure," Arch. Mech. Stosow., 19, No. 1 (1967).

MAXIMUM STABILITY FORMULAS FOR REINFORCED CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

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The problem of determining the reinforcement structure that maximizes the stability of a cylindrical shell subjected to external pressure was formulated in [1], where a numerical solution was obtained for a particular class of structures on the basis of a formula for the stability limit of a hinged anisotropic circular cylindrical shell of medium length in the membrane state. In the present article the stability limit is determined more accurately, without any constraint on the length of the shell, and the optimization is carried out over a broader class of structures.

§1. Let us consider a circular cylindrical shell of constant thickness H, mean radius R, and length L, made of fibrous composite material. It is assumed that the material has a regular layered structure, so that it is possible to distinguish a typical layer whose thickness is small as compared with that of the shell; the typical layer has multidirectional reinforcement symmetrical with respect to an arbitrary axial section of the shell; the fibers in all directions are made of the same linear-elastic material; the matrix material is linear-elastic and isotropic.

In order to describe the state of stress and strain of the typical layer, we will employ the mechanical model proposed in [2]. Under the assumptions formulated above, this model substitutes for an element of the reinforced layer the statically equivalent element of an orthotropic-elastic homogeneous layer whose state of stress and strain is determined in the principal surface coordinate system by the symmetric plane tensors of the average stresses σ_{ij} and strains ε_{ij} (the subscripts i, j run through the values 1, 2). The equations of [2] for the relationship between the components of these tensors, simplified in accordance with the starting assumptions, take the form

$$\sigma_{11} = \omega E(a_{11}\varepsilon_{11} + a_{12}\varepsilon_{12}),\ \sigma_{12} = \omega E a_{33}\varepsilon_{12},\ 1 \rightleftarrows 2; \tag{1.1}$$

$$a_{11} = \varepsilon + \sum_{k=1}^{K} \frac{\omega_k}{\omega}\chi_{1k}^2, \quad a_{12} = \varepsilon\nu_0 + \sum_{k=1}^{K}\frac{\omega_k}{\omega}\chi_{1k}^2\chi_{2k}^2, \quad 1 \rightleftarrows 2, \tag{1.2}$$

$$a_{33} = \varepsilon(1-\nu_0) + 2\sum_{k=1}^{K}\frac{\omega_k}{\omega}\chi_{1k}^2\chi_{2k}^2, \quad \omega_k \geqslant 0, \quad \omega = \sum_{k=1}^{K}\omega_k < 1,$$

$$\varepsilon = (1-\omega)E_0/(1-\nu_0^2)E\omega > 0, \quad 0 \leqslant \nu_0 \leqslant 1/2,$$

where E_0 and E are the moduli of elasticity of the matrix and the fibers, respectively; ν_0 is the Poisson ratio of the matrix; ω_k(k = 1, 2, ..., K) is the volume fraction of fibers of direction k (K is the total number of directions); ω is the volume fraction of reinforcement; χ_{ik} are the direction cosines of the k-th direction with respect to the i-th coordinate line.

For shell strains satisfying the Kirchhoff kinematic hypotheses we can write

$$\varepsilon_{ij} = p_{ij} + \zeta q_{ij},$$

where p_{ij}, q_{ij} are the symmetric tensors of the tangential and bending strains of the middle

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